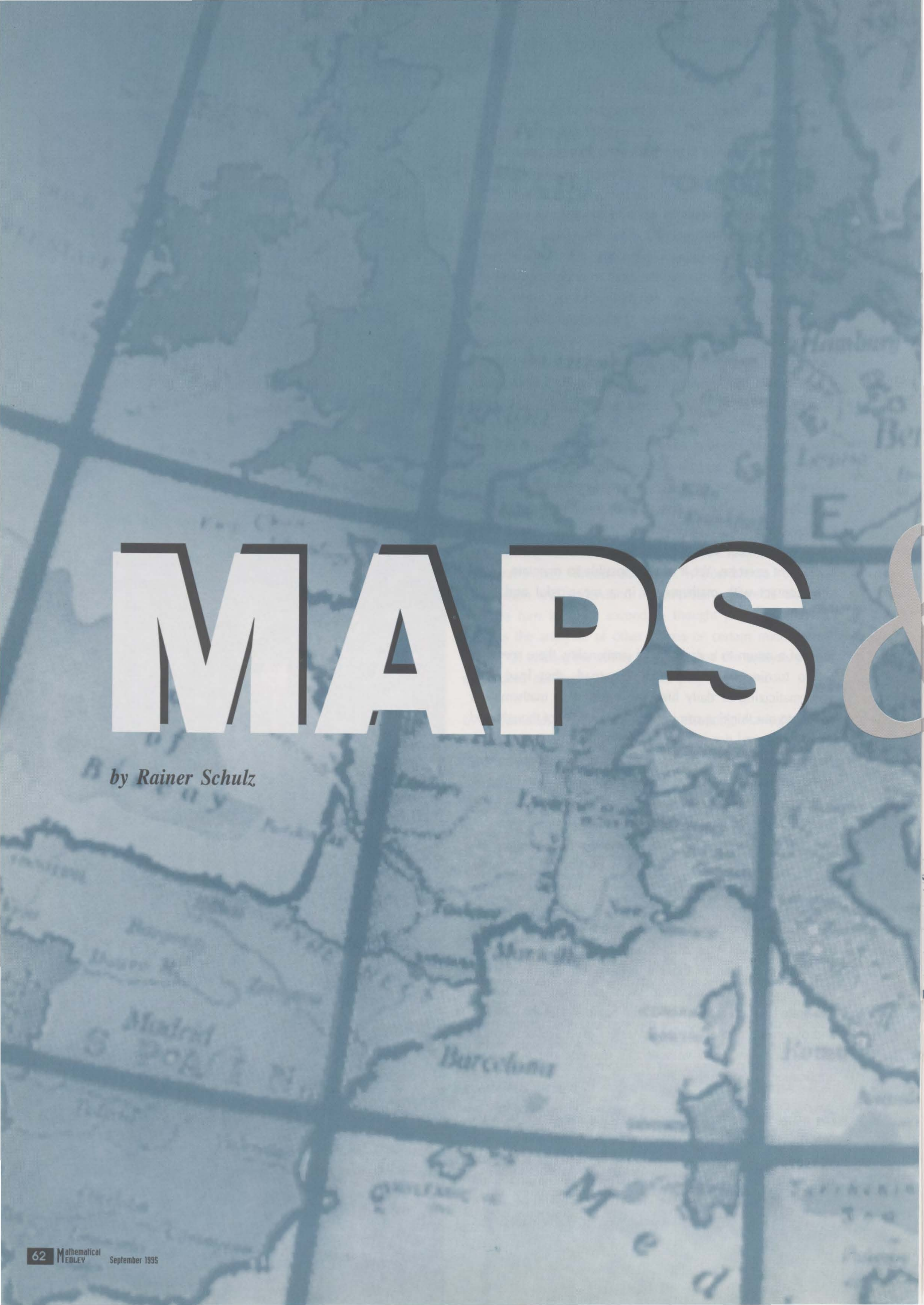

MAPS &

by Rainer Schulz

This article is based on a talk delivered to Secondary School students taking part in the Gifted Education Programme Science Camp 1995 at the National University of Singapore. It deals with the application of Mathematics to Geography, more precisely to the subject of map projections. The maps which appear in this article have been produced by some small BASIC programs. A diskette containing these programs will be given to any school which places an order of the current issue of the Medley. The interested reader may copy the files of this diskette and experiment with map projections on her or his own. See section 5 for details about the sample programs.

1. Introduction

Map making seems to be so easy! At first, fix any point on earth (your home, or, if you are on the payroll of a ministry of the current world super empire, preferably the center of the empire). Then, draw a straight line in North-South direction through that point, and call it the degree zero meridian. Now, draw a line which is orthogonal to the first one. You get a rectangular coordinate system. Secondly, draw another rectangular coordinate system on a sheet of paper, and, using some scale factor, transfer every point from nature to the paper.

Such a world map exists since long, it can be found in school books on History and is attributed to Eratosthenes (276-196), a Greek geographer who worked in Alexandria in Egypt. Not surprisingly, Eratosthenes has chosen this town as the centre of his coordinate system.

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If one would apply Eratosthenes' method today, one would perhaps choose the equator of the earth as a base line, and one would draw meridians and latitude lines in equal distances. This would create our first map projection which is called a cylindrical equidistant projection, or the Eratosthenes projection, for short.

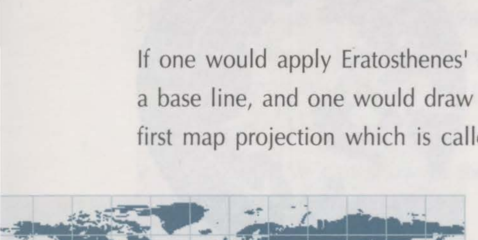


Fig. 1

The Eratosthenes projection (Fig. 1) is simple and looks just perfect, but it has a fundamental inaccuracy: The meridians are parallel lines only on the map. In reality, they are not, but they intersect in the North Pole and in the South pole. A lot more is wrong with the map. If a sailor would use it for navigation, and steer a constant compass course on sea, and plot his trip on the map, then he would not obtain a straight line, but a pretty complicated curve. If a sociologist would study the density of population and plot every person as a dot on the map, then the resulting picture would not properly reflect population density, since the sizes of areas are distorted.

Since the surface of the earth is a sphere, or globe, there will be no plane map without distortion. However, by choosing an appropriate map projection, one can make sure the map meets certain demands of the user, e.g. it *does* map constant compass course journeys as straight lines, or it *does* map the sizes of areas without distortion.

It is here where Mathematics enters the scene. What we need is a precise concept of "distortion". In order to control the distortion of a map, we have to describe the map by a formula. The next sections deal with the question of how to find these formulas.

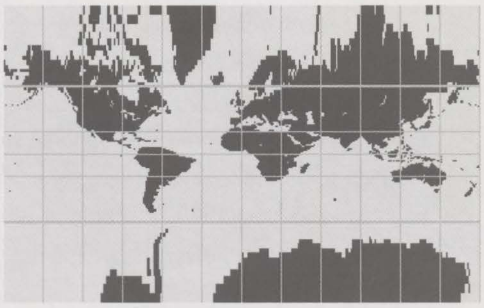


Fig. 2

2. Geometrically generated maps

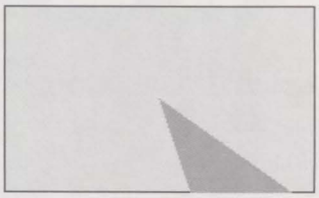
It is quite obvious how the Eratosthenes projection map (Fig. 1) can be described by a formula: The point *P* on the globe located at longitude λ and latitude ϕ is assigned to the point *P'* on the map with coordinates $x = c\lambda$ and $y = c\phi$. In this formula, *c* is some scaling factor. *For simplicity, we will omit the factor c in all formulas to follow.* Also, one may use the convention that λ ranges between -180 and 180 degrees (0 at Greenwich, negative to the West, positive to the East), and ϕ ranges between -90 and 90 degrees (0 at the equator, negative to the South, positive to the North).

There are three immediate methods to obtain maps geometrically. (The reader who is familiar with basic trigonometry may verify the given formulas.)

1) Wrap a plane around the globe such that it forms a cylinder which touches the globe at the equator. For any point *P* on the surface of the globe (well, almost any, one has to exclude the poles.), draw a straight line from the centre *C* of the globe through *P* until it intersects the cylinder in the point *P'*. The map from *P* to *P'* establishes the "cylindrical projection". Choosing a proper *x* - *y* - coordinates system on the plane, this map can be described by the formula $x = \lambda$ and $y = \tan\phi$,

on the PC

where tan denotes the tangent function.



2) Have a plane and cut a wedge-shaped piece off. Form what remains to a cone and let it rest on the top of the globe. Get the "conical projection" analogous to the procedure described in 1). The reader may try to find the formula for this projection by her or his own.

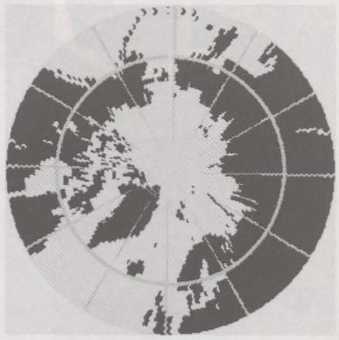


Fig. 3

3) Let a tangent plane rest on top of the globe. Again, apply the method from 1) to get an "azimuthal projection". If one uses polar coordinates, its formula is $\theta = \lambda$, $r = \cot\phi$. You may convert these into Cartesian coordinates.

It is obvious that the cylindrical projection is good for equatorial regions, but it vastly exaggerates sizes of areas close to the pole. Greenland looks bigger than South America, and Antarctica looks like the biggest of all continents (See Fig. 2). The azimuthal projection (Fig. 3) is fine for the polar regions, but it is unusable for regions close to the equator (though most favourable for Singapore...). The conical projection seems to be good for regions of medium latitude. In fact, most maps of the United States make use of this projection.


Fig. 5


Fig. 4

3. More Maps

There are many modifications of the map making methods described in section 2. One may use secant instead of tangent cylinders or cones. Also, one may choose different aspects, e.g. one may place a tangent plane not at the North pole, but at any point of the globe. Fig. 4 demonstrates how unusual an ordinary cylindric projection looks like, when the wrapping cylinder contacts the globe not at the equator, but at another great circle, e.g. the one which passes through Singapore and, say, Munich (Bavaria).

Although not generated in a geometric way, Eratosthenes' projection is also considered to be cylindrical, since it shows a grid pattern similar to that of the cylindrical projection.

The same holds for the famous Mercator projection, given by the formulas $x = \lambda$, $y = \ln(\tan(45 + {}^{\phi}/_{2}))$. It was invented by Gerhard Mercator in 1569 and is in use for maritime navigation up to the present day, since it just has the nice property mentioned in section 1 to map straight compass courses as straight lines.

What would be a good map of the moon? Since it doesn't happen too often that we set our foot on the moon's soil and have to navigate there, it may better meet astronomer's demands to map the moon as it appears in a telescope. This is done by the orthographic projection. See Fig. 5 for the Earth's view, under this projection. (How can this projection be constructed geometrically?)

The azimuthal equidistant projection with aspect North pole (Fig. 6) looks similar to the azimuthal projection from section 2. However, it is not constructed geometrically, but it is characterized by the property that distances from the North pole to any other point on the globe are shown in correct proportions. Check the formula $\theta = \lambda$, $r = 90 - \phi$, in polar coordinates.

How about maps which show area sizes in their true proportions? Surprisingly, they do exist, and there are many of them. They are called equal-area maps. The cylindrical equal-area projection is given by the formulas $x = \lambda$, $y = \sin\phi$. See Fig. 7. Note that, compared to Erastosthenes' projection, polar regions are vertically compressed, now. Another equal-area projection is the sinusoidal projection, $x = \lambda\cos\phi$, $y = \phi$, Fig. 8 We can show in the next section that these are in fact equal area projections.

My personal favourite is the Mollweide projection. It is also equal-area, and, in its ordinary aspect, it is often used for thematic world maps such as population maps or climate maps, see Fig. 9. A Singapore aspect is shown in Fig. 10. The Mollweide projection develops its entire beauty under oblique aspects, see the grid only in Fig. 11.

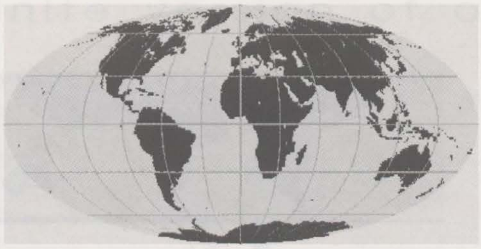


Fig. 9

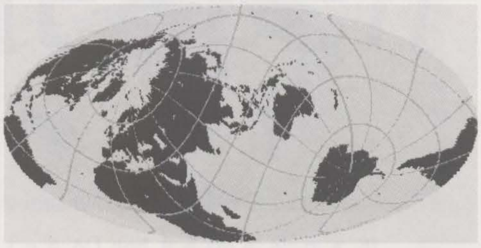


Fig. 10

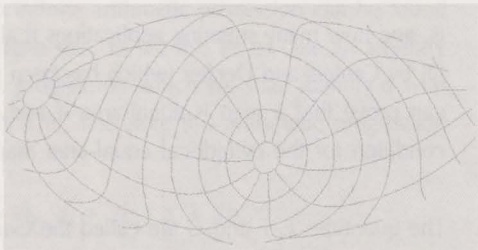


Fig. 11


Fig. 6

4. Distortion

Every map has a nominal scale, such as 1:100000. Does this mean that 1cm on the map represents 1km in nature? Not at all! This may only hold for certain points and in certain directions on the map. In other points of the map, 1cm may mean more or less than 1km. We can now give a definition of distortion. Imagine you are at a point *P* on the globe and looking into the direction α (say, $\alpha = 0$ means North, $\alpha = 45$ means Northeast etc.) Now draw a little arrow of length *l* on the ground, starting at *P* and having the direction of α. Find this arrow on your map and measure its length

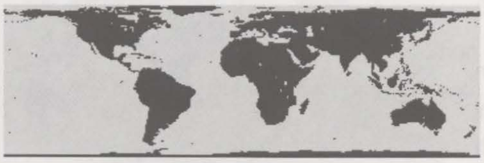


Fig. 7

Fig. 8

there. Multiply it by the nominal map scale to get the number L. Then the quotient $m = {}^1/_L$ is called the distortion of the map at P in the direction of α. Recall from sections 2 and 3 that every map can be described by a pair of formulas $x = f(\lambda, \phi)$ and $y = g(\lambda, \phi)$. Readers who are familiar with differentiability will be able to form f_λ, f_ϕ, g_λ, g_ϕ which simply mean the derivatives of the functions f and g with respect to the variables λ and ϕ, respectively. Now let $E = f_\phi^2 + g_\phi^2$. $F = f_\lambda f_\phi + g_\lambda g_\phi$, $G = f_\lambda^2 + g_\lambda^2$ (all of them evaluated at the point P). Then there is the following formula for m at P in the direction of α:

$$m^2 = E\cos^2\alpha + \frac{2}{\cos\phi} F\sin\alpha\cos\alpha + \frac{1}{\cos^2\phi} G\sin^2\alpha$$

It cannot be enough emphasized how useful this formula is, and how many stunning applications it allows. The interested reader is referred to the beautiful book [C-D] by Canters and Decleir (which has been very inspiring for me). Here, let us give one example only: One can prove that a map is equal-area if and only if $\sqrt{EG - F^2} = \cos\phi$. It is an easy exercise to verify this condition for the cylindrical equal-area map and the sinusoidal map from section 3!

The quantities E, F and G are called the Gaussian fundamental quantities, named after Carl Friedrich Gauβ (1777-1855). Gauβ is not only considered to be one of the greatest mathematicians ever, he also made far-reaching contributions to physics and astronomy. His work on map projections can be seen as a root of an area of mathematics which is most fashionable today: Differential Geometry. His country has recently honoured him by showing his portrait and symbols of some of his ideas on a banknote. (Or should one better say: The banknote has been decorated with ...?) You will find parts of the banknote reproduced in this article. On the title page of this edition of the "Medley", you can find Gauβ' portrait (with kind permission of Deutsche Bundesbank, Frankfurt am Main).

5. Maps visualized on the PC

All the maps in this article have been produced with the BASIC programs from the accompanying diskette. There are some more programs on the diskette, e.g. one to produce the Mercator map, and a movie of the rotating globe. Many of the programs allow the user to define the mapping aspect and to produce unusual maps which cannot be found in atlases.

Feel free to copy the files from the diskette, but be warned: The ad-hoc programs are experimental, not documented and by no means perfect! Everybody is encouraged to make improvements. Readers who are familiar with (spherical) trigonometry and with matrices may write their own programs, using the Eratosthenes map (this is on the file K.MAP) as a data base. Lucky readers who have access to the Internet will find many, many more data bases there, such as incredibly detailed pixel maps, contour maps, digital elevation data which would allow the creation of colourful topographic maps and much more. A first project might be the production of conical maps whose formulas have not been given in this article. There are endlessly many more projects. M²

Reference

[C-D] Frank Canters and Hugo Decleir, The world in perspective, Chichester 1989.

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